ON THE IONISATION IN THE NEBULAR ENVELOPE SURROUNDING A STAR.

V. A. Ambarzumian.

(Communicated by E. A. Milne)

At present there is scarcely any doubt that the emission lines in the spectra of Be stars, as well as the Wolf-Rayet emission bands, have their origin in extensive and tenuous atmospheres which form some sort of nebular envelope surrounding these stars. Recently some observational data about the emission lines and bands and their intensities have been collected by several observers. The correct theoretical interpretation of these data will be possible only in terms of the ionisation theory for such extended atmospheres. No such theory exists at present. An attempt to give the first outline of such a theory is given in the present paper.

It is clear that the ionisation depends chiefly upon the density of the corresponding "ultra-violet" radiation at the given point of the atmosphere. As soon as we have estimated this density our problem is approximately solved. But the calculation of this density is possible only on the basis of the theory of radiative equilibrium in these atmospheres. Thus the problem of ionisation is essentially a problem of the theory of radiative equilibrium.

A theory of radiative equilibrium of a planetary nebula was proposed by the present writer in two previous papers.* However, there is an important difference between the types of the radiative equilibrium in planetaries and in the nebular envelope of relatively small radius surrounding the stars with emission lines.

In a few words this difference may be explained in the following way. In the previous papers we have shown that the transformation of the quanta of the ultra-violet continuous spectrum into the quanta of the resonance line of hydrogen atom (L_α) on the one hand, and the large optical thickness of planetaries in the resonance frequency on the other, lead to a very high value of the density of L_α-radiation in the inner parts of a planetary nebula. Nevertheless, though exceeding 10^9 times the density of the direct L_α-radiation from the star falling on the inner boundary of nebulæ, this density is still many thousand times smaller than the density of L_α-radiation at the surface of the central star, since the dilution factor W is of the order of 10^{-13}. Therefore the relative number of atoms in the second level is also small compared with that on the surface of the star, and we may still neglect the transitions from the second level to the higher levels.

We know indeed that the number of such transitions per second is proportional to $n_2\rho_{23}$, where n_2 is the number of atoms per c.c. in the second

* *M.N.*, **93**, 50, 1932; *Bull. de l'observatoire central à Poulkovo*, **13**, 3, 1933.

level and ρ_{23} is the density of the corresponding radiation. For n_2 we have approximately $n_2 \cong n_1 \frac{\rho_{12}}{\sigma_{12}}$, where ρ_{12} is the density of radiation corresponding to transition $1 \rightarrow 2$ and $\sigma_{12} = \frac{8\pi h \nu_{12}^3}{c^3}$. Therefore the number of transitions $2 \rightarrow 3$ is proportional to $n_1 \bar{\rho}_{12} \rho_{23}$, while the number of transitions $1 \rightarrow 3$ is proportional to $n_1 \rho_{13}$. The coefficient of proportionality is of the same order of magnitude in both cases.

Now on the surface of the star we have approximately

$$(\bar{\rho}_{13})_{\text{surf}} = (\bar{\rho}_{12})_{\text{surf}} (\bar{\rho}_{23})_{\text{surf}}. \tag{A}$$

In the nebula we have $(\bar{\rho}_{13})_{\text{neb}} \cong W(\bar{\rho}_{13})_{\text{surf}}$, $(\bar{\rho}_{23})_{\text{neb}} = W(\rho_{23})_{\text{surf}}$ and $(\bar{\rho}_{12})_{\text{neb}} = \epsilon(\bar{\rho}_{12})_{\text{surf}}$, where ϵ is a small quantity of the order of 10^{-4} as was mentioned above.

Therefore taking into account (A) we find

$$(\bar{\rho}_{12})_{\text{neb}} (\bar{\rho}_{23})_{\text{neb}} \cong \epsilon (\bar{\rho}_{13})_{\text{neb}},$$

or

$$n_1 (\bar{\rho}_{12})_{\text{neb}} (\rho_{23})_{\text{neb}} \cong n_1 \frac{\nu_{23}^3}{\nu_{13}^3} \epsilon (\rho_{13})_{\text{neb}},$$

and we see that the number of the transitions $2 \rightarrow 3$ is a negligible fraction of the number of the transitions $1 \rightarrow 3$. At the same time a considerable portion of atoms arriving at the third state passes spontaneously to the second state, and we may say, therefore, that the number of transitions $2 \rightarrow 3$ is negligible compared with the number of transitions $3 \rightarrow 2$. In other words, we may neglect the number of the cyclic transitions of the type $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ compared with the number of the cyclic transitions of the type $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$.

In accordance with this, the method developed in our previous papers where the transitions $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ were neglected, is applicable to the planetaries.

It cannot be applied, however, to the cases where the dilution of radiation is not so high as in planetaries. Thus it is in no case applicable to the gaseous envelopes of the Wolf-Rayet stars and Be stars. In the Wolf-Rayet stars the envelopes are immediately attached to the surfaces of the stars and the maximum of density is at the surface itself. In this case, for example, W is of the order of unity, and our method is certainly not applicable. At the same time we cannot restrict ourselves in this case to the consideration of monochromatic radiative equilibrium, as we do in the case of the reversing layers of stars for the resonance lines, making use of the fact that the cyclic transitions are relatively rare when compared with the transitions of the type $1 \rightarrow 2 \rightarrow 1$. In fact, though this is still true in the case of Wolf-Rayet stars, we cannot neglect the cyclic transitions, since these transitions only are responsible for the appearance of the emission lines (or bands).

At first sight it seems questionable whether it is necessary to take into account the partial compensation of the "direct" cyclic processes, corresponding to the fluorescence phenomenon ($1 \to 3 \to 2 \to 1$), by reverse processes ($1 \to 2 \to 3 \to 1$), when we are convinced that our theory would have only a qualitative character owing to many other approximations of a physical and geometrical nature. It seems to us that for the question of the final intensities of the emission lines, it is of little difference whether we take into account the reverse processes or not. But if our purpose is the calculation of the ionisation in the envelopes, the difference between these two cases is very large. The exact formulæ we shall give below, but even now we may see that the degree of ionisation, which is proportional to the density of corresponding radiation, in the case of the complete compensation of the processes of both kinds can be represented according to Schuster's theory in the form :

$$\frac{n^+}{n} n_e = C \frac{t + 0{\cdot}5}{t_1 + 1}, \tag{B}$$

where t is the optical depth of the given point of the envelope and t_1 the whole optical thickness, while in the case when each absorbed "ultra-violet" quantum is split up into quanta of small frequencies and no reverse process occurs, we shall have

$$\frac{n^+}{n} n_e = C' e^{-(\tau_1 - \tau)}. \tag{C}$$

In the case when the optical thickness of the envelope is large compared with unity, the formulæ (B) and (C) give us the ionisation of quite a different order of magnitude. It is possible that the exact formula will give something between (B) and (C). But the difference between (B) and (C) is too large to give us any possibility even of approximate estimation of the quantity under consideration.

In the present paper we are performing some calculations which, perhaps, may serve this purpose.

The Conditions of the Steady State.—We shall consider an atom in which the electron has only three energy levels $\epsilon_1 < \epsilon_2 < \epsilon_3$. Each of these levels has the corresponding weight g_1, g_2, g_3. If the density of matter is so low that we may neglect the super-elastic collisions, and each electronic transition is accompanied by the corresponding radiation of a light quantum, the conditions of the steady state may be written in the form :

$$\left.\begin{aligned} n_1\{B_{12}\rho_{12} + B_{13}\rho_{13}\} &= n_2\frac{g_1}{g_2}B_{12}(\sigma_{12} + \rho_{12}) + n_3\frac{g_1}{g_3}B_{13}(\sigma_{13} + \rho_{13}) \\ n_1 B_{13}\rho_{13} + n_2 B_{23}\rho_{23} &= n_3\left\{\frac{g_1}{g_3}B_{13}(\sigma_{13} + \rho_{13}) + \frac{g_2}{g_3}B_{23}(\sigma_{23} + \rho_{23})\right\} \end{aligned}\right\}, \tag{1}$$

where n_k is the number of atoms in kth state in c.c., ρ_{ik} is the density of radiation with frequency $\nu_{ik} = \frac{|\epsilon_i - \epsilon_k|}{h}$, B_{ik} is Einstein's probability coefficients of transition from lower state i to the higher state k.

The quantities σ_{ik} are connected with ν_{ik} and the universal-constants by means of relation

$$\sigma_{ik} = \frac{8\pi h \nu_{ik}^3}{c^3}. \tag{2}$$

The equations (1) replace the equation of the radiative equilibrium in the case when the conditions of thermodynamical equilibrium are not fulfilled.

If we suppose now that the third energy level is not a discrete one, but corresponds to the case when the atom is ionised and the electron is in a free state, we shall introduce instead of the Einstein probability coefficients $B_{1\to3}$ and $B_{2\to3}$ from one discrete state to another, some probability coefficients of the photo-electric transitions defined in an appropriate way. Thus the whole number of the ionisations of atoms per c.c. during an interval of time dt is equal to

$$(n_1 b_{1\to3}\rho_{13} + n_2 b_{2\to3}\rho_{23})dt, \tag{3}$$

where ρ_{13} is the specific density of radiation for minimal frequency ν_{13}, which is necessary for the ionisation of the normal atom, and ρ_{23} is the specific density for minimum frequency ν_{23}, which is necessary for the ionisation of an excited atom. Strictly speaking, the number of ionisations, from the first level for example, depends not only on the specific density in the frequency ν_{13}, but also on the specific densities for all frequencies which satisfy the inequality $\nu > \nu_{13}$. However, we may suppose that the *relative* distribution of energy beyond the frequency ν_{13} is determined by a single parameter T ("temperature"), and assume that the expression (3) is valid under the condition that $b_{1\to3}$ and $b_{2\to3}$ are dependent on this parameter.

The number of the spontaneous recombinations during the same time dt is equal to the expression

$$n_3(a_{3\to1} + a_{3\to2})n_e dt, \tag{4}$$

where n_3 is now the number of ionised atoms, and $n_3 a_{3\to1} dt$ (or $n_3 a_{3\to2} dt$) is the probability of such recombination of a free electron with an ionised atom, which gives an atom in the first level (or in the second level). The simple considerations connected with the process of ionisation show that between our *a–s* and *b–s* there are the following relations :—

$$a_{3\to1} = \frac{g_1\sigma_{13}}{g^+G} b_{1\to3}\,; \qquad a_{3\to2} = \frac{g_2\sigma_{23}}{g^+G} b_{2\to3}, \tag{5}$$

where g^+ is the weight of the normal state of the ionised atom and

$$G = \frac{(2\pi m \kappa T)^{3/2}}{h^3}. \tag{6}$$

The expressions (3) and (4) for the number of transitions and the relations (5) and (6) make it possible for us to write in our case the conditions of the steady state in the form :

$$\left.\begin{aligned} n_1\{B_{12}\rho_{12}+b_{13}\rho_{13}\} &= n_2\frac{g_1}{g_2}B_{12}(\sigma_{12}+\rho_{12})+\frac{g}{g^+G}b_{13}(\sigma_{13}+\rho_{13})n_3n_e \\ n_1b_{13}\rho_{13}+n_2b_{23}\rho_{23} &= \frac{n_3n_e}{G}\left\{\frac{g_1}{g^+}b_{13}(\sigma_{13}+\rho_{13})+\frac{g_2}{g^+}b_{23}(\sigma_{23}+\rho_{23})\right\} \end{aligned}\right\}. \qquad (7)$$

The equations (7) may be brought to the form (1) if we denote in (7)

$$g_3=\frac{g^+G}{n_e} \qquad (8)$$

and write $B_{1\to 3}$ and B_{23} instead of b_{13} and b_{23}. The only difference is that the coefficients B_{13} and B_{23} as well as g_3 depend in our case on the parameter T. Therefore we can use the equations (1) as the starting-point of our calculations.

In the course of our discussions we shall suppose that B_{13}, B_{23} and g_3 are constant, since the range of the relative variation of these quantities is undoubtedly very small compared with the range of variation of other quantities entering into our investigation. This circumstance will have the consequence that we may obtain from our theory the information, not about the *distribution* of the "ultra-violet" radiation in the spectrum, but only about the approximate mean value of its intensity.

The Approximate Form of the Solution of the Equations of the Steady State.—Each specific density ρ_{ik} we may write in the form $\rho_{ik}=\bar\rho_{ik}\cdot\sigma_{ik}$. The quantities $\bar\rho_{ik}$ are dimensionless. According to Planck's law, at the surface of stars we shall have

$$\bar\rho_{ik}=\frac{1}{e^{\frac{h\nu_{ik}}{kT}}-1}.$$

Let us suppose that $\nu_{12}>2kT$ and $\nu_{23}>2kT$.* Then the quantities $\bar\rho_{ik}$ are even at the surface of the star small compared with unity. We have approximately at the surface $\bar\rho_{13}=\bar\rho_{12}\bar\rho_{23}$, and therefore if we call $\bar\rho_{12}$ and $\bar\rho_{23}$ small quantities of the first order, $\bar\rho_{13}$ will be of the second order. In the envelope $\bar\rho_{ik}$ may scarcely attain values many times exceeding their surface values. Therefore we may always regard them as small quantities, and especially $\bar\rho_{13}$ as a small quantity of the second order.

The equations of the transfer of radiation for the frequencies ν_{12} and ν_{13}, which we shall consider in the following paragraph, contain the following two expressions :—

$$\frac{\frac{g_1}{g_2}n_2}{n_1-\frac{g_1}{g_2}n_2} \quad \text{and} \quad \frac{\frac{g_1}{g_3}n_3}{n_1-\frac{g_1}{g_3}n_3}.$$

These ratios are to be calculated from the equations (1). The corresponding expressions are somewhat complicated. But they attain extraordinary simplicity if we restrict ourselves to the members of the first and

* These conditions are fulfilled almost in every case, which is of practical interest.

of the second order and neglect the higher terms. The calculations are too long to be reproduced here. The results are

$$\frac{\frac{g_1}{g_2}n_2}{n_1-\frac{g_1}{g_2}n_2}=\bar{\rho}_{12}+\gamma(\bar{\rho}_{13}-\bar{\rho}_{12}\bar{\rho}_{23}), \tag{9}$$

$$\frac{\frac{g_1}{g_3}n_3}{n_1-\frac{g_1}{g_3}n_3}=\bar{\rho}_{13}+\beta(\bar{\rho}_{12}\bar{\rho}_{23}-\bar{\rho}_{13}), \tag{10}$$

where the constants γ and β have the following values :—

$$\gamma=\frac{g_2B_{13}B_{23}\sigma_{13}\sigma_{23}}{B_{12}\sigma_{12}[g_1B_{13}\sigma_{13}+g_2B_{23}\sigma_{23}]}, \tag{11}$$

$$\beta=\frac{g_2B_{23}\sigma_{23}}{g_1B_{13}\sigma_{13}+g_2B_{23}\sigma_{23}}=\frac{B_{12}\sigma_{12}}{B_{13}\sigma_{13}}\gamma. \tag{12}$$

The approximation which we have applied has a very simple physical meaning. We may consider indeed the atomic transitions of the type $1\rightarrow 2\rightarrow 1$ as simple processes of the scattering of the quanta with the frequency ν_{12} by normal atoms. The more complicated processes of the type $1\rightarrow 2\rightarrow 3\rightarrow 1$ or $1\rightarrow 2\rightarrow 3\rightarrow 2\rightarrow 1$ we may consider as the processes of the collision of the normal atom with two quanta $h\nu_{12}$ and $h\nu_{23}$. After the collision of such type we may have either one quantum with the frequency ν_{13}, or again two quanta $h\nu_{12}$ and $h\nu_{23}$. There may also occur the collisions of three or more quanta with a normal atom. Such are, for example, the processes of the type $1\rightarrow 2\rightarrow 3\rightarrow 2\rightarrow 3\rightarrow 1$, where two quanta $h\nu_{23}$ and a quantum $h\nu_{12}$ collide with the normal atom. If, however, the density of radiation for every frequency is small we may neglect all collisions where more than two quanta take part. Assuming further that the density of radiation of the frequency ν_{13} is small even when compared with ρ_{12} and ρ_{23}, we may neglect also such collisions with two quanta at once, when at last one of two quanta has the frequency ν_{13}. Thus we shall obtain again the equations (9) and (10).

From this point of view the "classical" theory of monochromatic radiative equilibrium, which, as it was emphasised by some writers, is applicable only to the resonance lines, is the first approximation where only the small quantities of the first order are taken into account. This approximation is sufficient to explain on general lines the formation of absorption lines, but cannot give the explanation of the existence of the emission lines. We know (according to Rosseland) that such an explanation is impossible unless three frequencies and the cyclic transitions are taken into account. The theory outlined in the present paper is a second approximation, since the terms of the second order are taken into account. It has general character, while the theory given in the previous works of the writer is applicable only to the special cases when, owing to very strong dilution of radiation, we may neglect the term $\bar{\rho}_{12}\bar{\rho}_{23}$ compared with $\bar{\rho}_{13}$. The present theory, though still very far from accurate, will give in every case the main features of the phenomena of emission-lines.

The Equations of Transfer.—For each frequency appearing in our consideration we may write the corresponding equation of transfer. But it is superfluous to do it for the frequency ν_{23}, since we may suppose that the total optical thickness of the gaseous envelope in this frequency is certainly small compared with unity. In fact this frequency corresponds to the ionisation from an excited state, and therefore the optical thickness in it will be of the same order of magnitude as the optical thickness arising from the *general opacity* of the envelope, since this opacity is caused chiefly by such bound-free transitions from excited states. The optical thickness arising from the general opacity reaches the values of the order of unity only in the photosphere of the star. The same will be true for the frequency ν_{23}, and the envelope will be transparent for this frequency. Therefore we may write directly

$$\bar{\rho}_{23} = \frac{W}{e^{\frac{h\nu_{23}}{\kappa T}} - 1}, \tag{13}$$

where W is the coefficient of dilution.

The equations of the transfer of radiation for the frequencies ν_{12} and ν_{13} as it is known * may be written in the form :

$$\frac{dI_{12}}{\frac{h\nu_{12}}{\Delta\nu_{12}}\left(n_1 - \frac{g_1}{g_2}n_2\right)\frac{B_{12}}{c}ds} = -I_{12} + \frac{g_1}{g_2}\frac{c\sigma_{12}}{4\pi}\frac{n_2}{n_1 - \frac{g_1}{g_2}n_2}, \tag{14}$$

$$\frac{dI_{13}}{\frac{h\nu_{13}}{\Delta\nu_{13}}\left(n_1 - \frac{g_1}{g_3}n_3\right)\frac{B_{13}}{c}ds} = -I_{13} + \frac{g_1}{g_3}\frac{c\sigma_{13}}{4\pi}\frac{n_3}{n_1 - \frac{g_1}{g_3}n_3}, \tag{15}$$

where I_{12} and I_{13} are the specific intensities of radiation with frequencies ν_{12} and ν_{13}, ds is the element of the trajectory of the ray, $\Delta\nu_{12}$ is the width of the resonance line and $\Delta\nu_{13}$ the effective width of the corresponding absorption line.

If we introduce the absorption coefficient α per unit of volume for the frequency ν_{12},

$$\alpha = \frac{h\nu_{12}}{\Delta\nu_{12}}\left(n_1 - \frac{q_1}{q_2}n_2\right)\frac{B_{12}}{c}, \tag{16}$$

we obtain

$$\frac{dI_{12}}{\alpha ds} = -I_{12} + \frac{g_1}{g_2}\frac{c\sigma_{12}}{4\pi}\frac{n_2}{n_1 - \frac{g_1}{g_2}n_2}, \tag{17}$$

$$\frac{dI_{13}}{\alpha ds} = \frac{\nu_{13}\Delta\nu_{12}}{\nu_{12}\Delta\nu_{13}}\frac{n_1 - \frac{g_1}{g_3}n_3}{n_1 - \frac{g_1}{g_2}n_2}\frac{B_{13}}{B_{12}}\left\{-I_{13} + \frac{g_1}{g_3}\frac{c\sigma_{13}}{4\pi}\frac{n_3}{n_1 - \frac{g_1}{g_3}n_3}\right\}. \tag{18}$$

* Milne, *M.N.*, **88**, 493, 1928.

It is convenient to introduce instead of intensities I_{12} and I_{13} the dimensionless quantities ϕ and ψ:

$$I_{12}=\frac{c\sigma_{12}}{4\pi}\psi; \qquad I_{13}=\frac{c\sigma_{13}}{4\pi}\psi. \tag{19}$$

Then the equations (17) and (18) obtain the simple form:

$$\frac{d\phi}{ads}=-\phi+\frac{\frac{g_1}{g_2}n_2}{n_1-\frac{g_1}{g_2}n_2}, \tag{20}$$

$$\frac{d\psi}{ads}=\frac{\nu_{13}\Delta\nu_{12}}{\nu_{12}\Delta\nu_{13}}\frac{n_1-\frac{g_1}{g_3}n_3}{n_1-\frac{g_1}{g_2}n_2}\frac{B_{13}}{B_{12}}\left\{-\psi+\frac{\frac{g_1}{g_3}n_3}{n_1-\frac{g_1}{g_3}n_3}\right\}. \tag{21}$$

It is clear that between ϕ and ψ from one side and $\bar{\rho}_{12}$ and $\bar{\rho}_{13}$ from another we have the following relations :—

$$\bar{\rho}_{12}=\frac{1}{4\pi}\int\phi d\omega; \qquad \bar{\rho}_{13}=\frac{1}{4\pi}\int\psi d\omega, \tag{22}$$

where $d\omega$ is the element of the solid angle and the integration is carried over all directions.

If $\bar{\rho}_{13}$ is a small quantity of the second order, the quantity ψ will be also of the second order. The quantity in brackets in (21) is then also of the second order, and in the factor before these brackets we may freely neglect the small quantities of the first order and write

$$\frac{n_1-\frac{g_1}{g_3}n_3}{n_1-\frac{g_1}{g_2}n_2}=1.$$

If we denote further,

$$\frac{\nu_{13}\Delta\nu_{12}}{\nu_{12}\Delta\nu_{13}}\frac{B_{13}}{B_{12}}=q,$$

we may write, taking into account (9), (10) and (22) instead of (20) and (21),

$$\frac{d\phi}{ads}=-\phi+\frac{1}{4\pi}\int[\phi+\gamma(\psi-\bar{\rho}_{23}\phi)]d\omega, \tag{23}$$

$$\frac{d\psi}{qads}=-\psi+\frac{1}{4\pi}\int[\psi+\beta(\bar{\rho}_{23}\phi-\psi)]d\omega. \tag{24}$$

It remains now to solve these equations, where ϕ and ψ are the functions of co-ordinates and of the direction of radiation. However, the solution may be actually performed only in the case when the boundary conditions for ϕ and ψ and the function a of co-ordinates are given.

The Geometrical Model.—The equations (23) and (24) have general character and are applicable to all atmospheric problems with any (constant or variable) coefficient of dilution. However, we exclude from further considerations the Wolf-Rayet stars owing to the fact that the high velocity of ejection of matter from them leads to an inequality of frequencies of the given line in different parts of the atmosphere. Therefore the problem in this case cannot be reduced to a one-dimensional one.

In the calculations given below we shall restrict ourselves again to the cases when the dilution factor is small compared with unity. Let us suppose, for example, that $W < \frac{1}{100}$. In this case we may with safety use the method of the reduction of the spherical problem to a plane problem developed by Professor Milne.* The only difference will be in the circumstance that we shall count the diffuse radiation and the direct radiation coming from the star together, while in Milne's work, as well as in the writer's papers, they were treated separately. This will make the method slightly less accurate. But the calculations are in this case not so complicated.

Let us introduce the optical depth at the distance r from the centre of the star :

$$\tau = \int_r^{r_2} \alpha dr,$$

where r_2 is the outer boundary of the nebular shell. If further r_1 is the distance of the inner boundary from the centre, we shall write

$$\tau_1 = \int_{r_1}^{r_2} \alpha dr.$$

τ_1 is the whole optical thickness of the shell.

Using the approximation of the Schwarzschild-Schuster type and introducing the average values ϕ and ϕ' of ϕ for the outward and inward directions of radiation and the corresponding average values ψ and ψ' of the quantity ψ, we have instead of (23) and (24) the following approximate equations :—

$$\frac{1}{2}\frac{d\phi}{d\tau} = \phi - \tfrac{1}{2}\{(\phi + \phi') + \gamma[\psi + \psi' - \bar{\rho}_{23}(\phi + \phi')]\}, \tag{25}$$

$$\frac{1}{2}\frac{d\phi'}{d\tau} = \tfrac{1}{2}\{(\phi + \phi') + \gamma[\psi + \psi' - \bar{\rho}_{23}(\phi + \phi')]\} - \phi', \tag{26}$$

$$\frac{1}{2q}\frac{d\psi}{d\tau} = \psi - \tfrac{1}{2}\{(\psi - \psi') + \beta[\bar{\rho}_{23}(\phi + \phi') - (\psi + \psi')]\}, \tag{27}$$

$$\frac{1}{2q}\frac{d\psi'}{d\tau} = \tfrac{1}{2}\{(\psi + \psi') + \beta[\bar{\rho}_{23}(\phi + \phi') - (\psi + \psi')]\} - \psi'. \tag{28}$$

At the outer boundary we have the following conditions :—

$$\phi'(0) = \psi'(0) = 0. \tag{D}$$

* E. A. Milne, *Zs. f. Astrophysik*, **1**, 98, 1930.

At the inner boundary we have for the resonance frequency the condition that the resulting flux vanishes,

$$\phi(\tau_1) - \phi'(\tau_1) = 0, \tag{E}$$

since we may neglect the direct radiation from the star in this frequency. For the "ultra-violet" radiation we have at this boundary

$$\psi(\tau_1) - \psi'(\tau_1) = \psi_0, \tag{F}$$

owing to the fact that $\psi(\tau_1)$ contains the direct radiation from the star. The constant ψ_0 is connected with the amount πS of the ultra-violet energy falling on each square centimetre of the inner surface by means of the relation:

$$\psi_0 = \tfrac{1}{2}\frac{4\pi S}{c\sigma_{13}} = 2W(\bar{\rho}_{13})_{\text{surf}}.$$

It remains now to solve the equations (25), (26), (27) and (28), taking into account the boundary conditions (D), (E) and (F).

The Solution of the Equations.—Adding (25) with (26) and (27) with (28) we obtain

$$\frac{1}{2}\frac{d(\phi+\phi')}{d\tau} = \phi - \phi', \tag{29}$$

$$\frac{1}{2q}\frac{d(\psi+\psi')}{d\tau} = \psi - \psi'. \tag{30}$$

Subtracting (26) from (25) and (28) from (27) we have

$$\frac{1}{2}\frac{d(\phi-\phi')}{d\tau} = -\gamma[\psi+\psi'-\bar{\rho}_{23}(\phi+\phi')], \tag{31}$$

$$\frac{1}{2q}\frac{d(\psi-\psi')}{d\tau} = -\beta[\bar{\rho}_{23}(\phi-\phi')-(\psi+\psi')]. \tag{32}$$

Differentiating (29) and (30) and comparing with (31) and (32) we obtain the following system of two second-order equations for $\phi+\phi'$ and $\psi+\psi'$:—

$$\frac{1}{4}\frac{d^2(\phi+\phi')}{d\tau^2} = -\gamma[\psi+\psi'-\bar{\rho}_{23}(\phi+\phi')], \tag{33}$$

$$\frac{1}{4q^2}\frac{d^2(\psi+\psi')}{d\tau^2} = \beta[\psi+\psi'-\bar{\rho}_{23}(\phi+\phi')]. \tag{34}$$

In Milne's model the linear thickness of the nebular shell is assumed small compared with the distance from the centre. Therefore it is consistent with this model to put $W=\text{const}$, or according to (13), $\bar{\rho}_{23}=\text{const}$.

In this case the general solution of the system of equations (33) and (34) has the form :

$$\left.\begin{aligned} \phi+\phi' &= C_1 + C_2\tau + C_3e^{\lambda\tau} + C_4e^{-\lambda\tau} \\ \psi+\psi' &= \bar{\rho}_{23}(C_1+C_2\tau) - \frac{q^2\beta}{\gamma}(C_3e^{\lambda\tau}+C_4e^{-\lambda\tau}) \end{aligned}\right\}, \tag{35}$$

where

$$\lambda = 2\sqrt{\gamma\bar{p}_{23} + \beta q^2}.$$

Using (29) and (30) we obtain now :

$$\left.\begin{aligned} \phi - \phi' &= \frac{C_2}{2} + \frac{C_3\lambda}{2}e^{\lambda\tau} - \frac{C_4\lambda}{2}e^{-\lambda\tau} \\ \psi - \psi' &= \frac{\bar{p}_{23}}{2q}C_2 - \frac{q\beta}{2\gamma}\lambda C_3 e^{\lambda\tau} + \frac{q\beta}{2\gamma}\lambda C_4 e^{-\lambda\tau} \end{aligned}\right\}. \tag{36}$$

Comparing (36) with (E) and (F) we have

$$C_2 + C_3\lambda e^{\lambda\tau_1} - C_4\lambda e^{-\lambda\tau_1} = 0, \tag{37}$$

$$\frac{\bar{p}_{23}}{2q}C_2 - \frac{q\beta}{2\gamma}\lambda C_3 e^{\lambda\tau_1} + \frac{q\beta}{2\gamma}\lambda C_4 e^{-\lambda\tau_1} = \psi_0. \tag{38}$$

From (35) and (36) we have further

$$2\phi' = \left(C_1 - \frac{C_2}{2}\right) + C_2\tau + C_3\left(1 - \frac{\lambda}{2}\right)e^{\lambda\tau} + C_4\left(1 + \frac{\lambda}{2}\right)e^{-\lambda\tau}, \tag{39}$$

$$2\psi' = \left(C_1 - \frac{C_2}{2q}\right)\bar{p}_{23} + C_2\bar{p}_{23}\tau - \frac{q^2\beta}{\gamma}\left[\left(1 - \frac{\lambda}{2q}\right)C_3 e^{\lambda\tau} + \left(1 + \frac{\lambda}{2q}\right)C_4 e^{-\lambda\tau}\right]. \tag{40}$$

Therefore the boundary conditions (D) may be written in the form :

$$\left(C_1 - \frac{C_2}{2}\right) + C_3\left(1 - \frac{\lambda}{2}\right) + C_4\left(1 + \frac{\lambda}{2}\right) = 0, \tag{41}$$

$$\left(C_1 - \frac{C_2}{2q}\right)\bar{p}_{23} - \frac{q^2\beta}{\gamma}\left[\left(1 - \frac{\lambda}{2q}\right)C_3 + \left(1 + \frac{\lambda}{2q}\right)C_4\right] = 0. \tag{42}$$

From the conditions (37), (38), (41) and (42) we can determine the coefficients C_1, C_2, C_3 and C_4. We have

$$\left.\begin{aligned} C_1 = {} & \frac{q\psi_0}{\bar{p}_{23}}\frac{1}{1 + \dfrac{q^2\beta}{\gamma\bar{p}_{23}}} \\ & - \left[\left(1 - \frac{\lambda}{2}\right)e^{-\lambda\tau_1} + \left(1 + \frac{\lambda}{2}\right)e^{\lambda\tau_1}\right]\frac{(q-1)\psi_0}{\bar{p}_{23}\left(1 + \dfrac{q^2\beta}{\gamma\bar{p}_{23}}\right)} \\ & - \frac{-\left\{\left[\left(1 + \frac{\lambda}{2}\right) + \frac{q^2\beta}{\gamma\bar{p}_{23}}\left(1 + \frac{\lambda}{2q}\right)\right]\left(1 - \frac{\lambda}{2}\right) - \left[\left(1 - \frac{\lambda}{2}\right) + \frac{q^2\beta}{\gamma\bar{p}_{23}}\left(1 - \frac{\lambda}{2q}\right)\right]\left(1 + \frac{\lambda}{2}\right)\right\}\dfrac{2q\psi_0}{\gamma\bar{p}_{23}}\dfrac{1}{1 + \frac{q^2\beta}{\gamma\bar{p}_{23}}}}{\left[\left(1 + \frac{\lambda}{2}\right) + \frac{q^2\beta}{\gamma\bar{p}_{23}}\left(1 + \frac{\lambda}{2q}\right)\right]e^{\lambda\tau_1} + \left[\left(1 - \frac{\lambda}{2}\right) + \frac{q^2\beta}{\gamma\bar{p}_{23}}\left(1 + \frac{\lambda}{2q}\right)\right]e^{-\lambda\tau_1}} \end{aligned}\right\} \tag{43}$$

$$\left.\begin{aligned} C_2 &= \frac{2q\psi_0}{\bar{p}_{23}} \frac{1}{1+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}} \\ C_3 &= -\frac{\dfrac{(1-q)\psi_0}{\bar{p}_{23}\left(1+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}\right)}e^{-\lambda\tau_1}+\left[\left(1+\dfrac{\lambda}{2}\right)+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}\left(1+\dfrac{\lambda}{2q}\right)\right]\dfrac{2q\psi_0}{\lambda\bar{p}_{23}}\dfrac{1}{1+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}}}{\left[\left(1+\dfrac{\lambda}{2}\right)+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}\left(1+\dfrac{\lambda}{2q}\right)\right]e^{\lambda\tau_1}+\left[\left(1-\dfrac{\lambda}{2}\right)+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}\left(1-\dfrac{\lambda}{2q}\right)\right]e^{-\lambda\tau_1}} \\ C_4 &= -\frac{\dfrac{(1-q)\psi_0}{\bar{p}_{23}\left(1+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}\right)}e^{\lambda\tau_1}+\left[\left(1-\dfrac{\lambda}{2}\right)+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}\left(1-\dfrac{\lambda}{2q}\right)\right]\dfrac{2q\psi_0}{\gamma\bar{p}_{23}}\dfrac{1}{1+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}}}{\left[\left(1+\dfrac{\lambda}{2}\right)+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}\left(1+\dfrac{\lambda}{2q}\right)\right]e^{\lambda\tau_1}+\left[\left(1-\dfrac{\lambda}{2}\right)+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}\left(1-\dfrac{\lambda}{2q}\right)\right]e^{-\lambda\tau_1}}. \end{aligned}\right\} \quad (43)$$

Ionisation in the Envelope.—Our purpose is to find the ionisation in the envelope. When the envelope is transparent to radiation of the frequency ν_{13}, the ionisation is to be calculated according to the simple formulæ given by Eddington and Rosseland. However, the most interesting case is that when the envelope is not transparent to this radiation, *i.e.* the optical thickness in the frequency ν_{13}, which is equal to $q\tau_1$, is large compared with unity. Thus $\tau_1 >> \frac{1}{q}$. Usually $\frac{1}{q}$ is of the order of 10^4. Therefore, if the dilution of radiation is not as high as in planetaries, the quantity $e^{-\lambda\tau_1}$ is exceedingly small and $e^{\lambda\tau}$ exceedingly large. This circumstance gives the possibility of simplifying the expressions for C_3 and C_4, neglecting small quantities. Thus we obtain from the formulæ given above the following approximate values :—

$$C_3 = \frac{2q\psi_0}{\lambda\bar{p}_{23}} \frac{1}{1+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}} e^{-\lambda\tau_1}; \qquad C_4 = -\frac{\dfrac{(1-q)\psi_0}{\bar{p}_{23}\left(1+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}\right)}}{\left(1+\dfrac{\lambda}{2}\right)+\dfrac{q^2\beta}{\gamma\bar{p}_{23}}\left(1+\dfrac{\lambda}{2q}\right)}. \quad (44)$$

It is clear now that the expression

$$C_3 e^{\lambda\tau} + C_4 e^{-\lambda\tau}$$

is always very small except near $\tau = 0$ and $\tau = \tau_1$. In all other parts of the envelope, practically speaking in the whole envelope, we may write, neglecting these terms, instead of (35):

$$\left.\begin{aligned} \phi + \phi' &= C_1 + C_2\tau \\ \psi + \psi' &= \bar{p}_{23}(C_1 + C_2\tau) \end{aligned}\right\}. \quad (45)$$

Let us fix our attention on the case when the dilution of radiation is not high and therefore $\bar{\rho}_{23} >> q^2$.* In this case we have simply

$$C_2 = \frac{2q\psi_0}{\bar{\rho}_{23}}, \tag{46}$$

since β and γ are approximately of the same order of magnitude.

Taking into account that λ is small compared with unity, we have also

$$C_1 \cong \frac{\psi_0}{\bar{\rho}_{23}}. \tag{47}$$

Therefore

$$\left.\begin{aligned} \phi + \phi' &\cong \frac{2\psi_0}{\bar{\rho}_{23}}(q\tau + \tfrac{1}{2}) \\ \psi + \psi' &\cong 2\psi_0(q\tau + \tfrac{1}{2}) \end{aligned}\right\}. \tag{48}$$

We may now obtain the approximate values of $\bar{\rho}_{12}$ and $\bar{\rho}_{13}$. We have

$$\bar{\rho}_{12} = \tfrac{1}{2}(\phi + \phi')\,; \qquad \bar{\rho}_{23} = \tfrac{1}{2}(\psi + \psi'),$$

and $\tau = \frac{t}{q}$ where t is the optical depth in the frequency v_{13}. Thus :

$$\left.\begin{aligned} \bar{\rho}_{12} &\cong \frac{\psi_0}{\bar{\rho}_{23}}(t + \tfrac{1}{2}) \\ \bar{\rho}_{13} &\cong \psi_0\,(t + \tfrac{1}{2}) \end{aligned}\right\}. \tag{49}$$

If we remember now that

$$\psi_0 = 2W(\bar{\rho}_{13})_{\text{surf}}\,; \qquad \bar{\rho}_{23} = W(\bar{\rho}_{23})_{\text{surf}}\,; \qquad (\bar{\rho}_{13})_{\text{surf}} = (\bar{\rho}_{12})_{\text{surf}}(\bar{\rho}_{23})_{\text{surf}},$$

we can write instead of (49)

$$\left.\begin{aligned} \bar{\rho}_{12} &\cong 2(\bar{\rho}_{12})_{\text{surf}}(t + \tfrac{1}{2}) \\ \bar{\rho}_{13} &\cong 2W(\bar{\rho}_{13})_{\text{surf}}(t + \tfrac{1}{2}) \end{aligned}\right\}. \tag{50}$$

We see now that if our assumptions are verified the density of radiation in the resonance frequency in the envelope is larger than the same density on the surface of the star. But further investigations are necessary to examine how far this important conclusion remains valid when the super-elastic collisions in the envelope are taken into consideration.

Introducing (50) in (10) we find the degree of ionisation :

$$\frac{\frac{g_1}{g_3}n_3}{n_1 - \frac{g_1}{g_3}n_3} \cong \bar{\rho}_{13} \cong W(\bar{\rho}_{13})_{\text{surf}} \cdot (2t + 1)\,;$$

or approximately :

$$\frac{g_1}{g_3}\frac{n_3}{n_1} \cong W(\bar{\rho}_{13})_{\text{surf}}(2t + 1). \tag{51}$$

* We have $\bar{\rho}_{23} = W(\bar{\rho}_{23})_{\text{surf}}$. Further, $q \cong 10^{-4}$. If $(\bar{\rho}_{23})_{\text{surf}}$ is of the order of 10^{-2}, the inequality in text is satisfied when $W >> 10^{-6}$. It is certainly not satisfied in the case of planetaries.